Worked conjugate full conditional distributions

– Perry Williams

Linear Regression: Normal likelihood
Normal Priors for β

$$\underline{y} \sim N(\underline{X}\underline{\beta}, \sigma^2 \underline{I})$$

$$\underline{\beta} \sim N(\underline{\mu}_\beta, \Sigma_\beta), \qquad \Sigma = \sigma_\beta^2 \underline{I}$$

$$\sigma^2 \sim IG(r, q)$$

complete the square.

Full conditional for $\underline{\beta}$

$$[\underline{\beta} \mid \cdot] \propto \exp\left\{-\frac{1}{2}(\underline{y} - \underline{X}\underline{\beta})'(\sigma^2 \underline{I})^{-1}(\underline{y} - \underline{X}\underline{\beta})\right\}$$

$$\times \exp\left\{-\frac{1}{2}(\underline{\beta} - \underline{\mu}_\beta)'(\sigma_\beta^2 \underline{I})^{-1}(\underline{\beta} - \underline{\mu}_\beta)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\underline{y}'(\sigma^2 \underline{I})^{-1} - \underline{\beta}'X'(\sigma^2 \underline{I})^{-1}\right)(\underline{y} - X\underline{\beta})\right.$$

$$\left. + \underline{\beta}'(\sigma_\beta^2 \underline{I})^{-1} - \underline{\mu}_\beta'(\sigma^2 \underline{I})^{-1}\right)(\underline{\beta} - \underline{\mu}_\beta)$$

$$\propto \exp\left\{-\frac{1}{2}\left(\underbrace{\underline{y}'(\sigma^2 \underline{I})^{-1}\underline{y}}_{F} - \underbrace{\underline{y}'(\sigma^2 \underline{I})^{-1}X\underline{\beta}}_{O}\right.\right.$$

$$- \underbrace{\underline{\beta}'X'(\sigma^2 \underline{I})^{-1}\underline{y}}_{I} + \underbrace{\underline{\beta}'X'(\sigma^2 \underline{I})^{-1}X\underline{\beta}}_{L}$$

$$+ \underbrace{\underline{\beta}'(\sigma_\beta^2 \underline{I})^{-1}\underline{\beta}}_{F} - \underbrace{\underline{\beta}'(\sigma_\beta^2 \underline{I})^{-1}\underline{\mu}_\beta}_{O}$$

$$\left.\left. - \underbrace{\underline{\mu}_\beta(\sigma_\beta^2 \underline{I})^{-1}\underline{\beta}}_{I} + \underbrace{\underline{\mu}_\beta'(\sigma_\beta^2 \underline{I})^{-1}\underline{\mu}_\beta}_{L}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[-2y'(\sigma^2 I)^{-1}X\beta + \beta' X'(\sigma^2 I)^{-1}X\beta - 2\mu_\beta(\sigma_\beta^2 I)^{-1}\beta + \beta'(\sigma_\beta^2 I)^{-1}\beta\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\Big(-2\underbrace{\left(y'(\sigma^2 I)^{-1}X + \mu_\beta'(\sigma_\beta^2 I)^{-1}\right)}_{b'}\beta + \left(\beta' X'(\sigma^2 I)^{-1}X + \beta'(\sigma_\beta^2 I)^{-1}\right)\beta\right.$$

$$\propto \exp\left\{-\frac{1}{2}\Big(-2\underbrace{\left(y'(\sigma^2 I)^{-1}X + \mu_\beta(\sigma_\beta^2 I)^{-1}\right)}_{b'}\beta + \beta'\underbrace{\left(X'(\sigma^2 I)^{-1}X + (\sigma_\beta^2 I)^{-1}\right)}_{A}\beta\right.$$

$$= N\left(A^{-1}b,\ A^{-1}\right)$$

$$= N\Big(\underbrace{\left[X'(\sigma^2 I)^{-1}X + (\sigma_\beta^2 I)^{-1}\right]^{-1}\left[y'(\sigma^2 I)^{-1}X + \mu_\beta'(\sigma_\beta^2 I)^{-1}\right]}_{\text{mean}},\ \underbrace{\left[X'(\sigma^2 I)^{-1}X + (\sigma_\beta^2 I)^{-1}\right]^{-1}}_{\text{VAR}}\Big)$$

$\square$

Linear Regression: Normal likelihood
IG Prior for σ^2

$$y \sim N(X\beta, \sigma^2)$$

$$\sigma^2 \sim IG(a, b)$$

Full conditional for σ^2

$$[\sigma^2 | \cdot] \propto |\sigma^2 I|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (y - X\beta)' (\sigma^2 I)^{-1} (y - X\beta) \right\}$$

$$\times \frac{b^a}{\Gamma(a)} (\sigma^2)^{-(a+1)} e^{-\frac{b}{\sigma^2}}$$

(note: $|aI| = a^n |I| = a^n 1$)

$$\propto (\sigma^2)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2} (y - X\beta)' I^{-1} (\sigma^2)^{-1} (y - X\beta) \right\}$$

$$\times (\sigma^2)^{-(a+1)} e^{-\frac{b}{\sigma^2}}$$

$$\propto (\sigma^2)^{-\left(\frac{n}{2} + a + 1\right)} \exp\left\{ -\frac{1}{2\sigma^2} \left[(y - X\beta)'(y - X\beta) + \frac{1}{b} \right] \right\}$$

$$\propto IG\left(\frac{n}{2} + a, \left[(y - X\beta)'(y - X\beta) + \frac{1}{b} \right] \right)$$

$\square$

N-mixture model full conditional distributions

-Perry Williams

Model

$$Y_{ij} \sim \text{BINOMIAL}(N_i, p)$$
$$N_i \sim \text{POISSON}(\lambda)$$
$$\lambda \sim \text{GAMMA}(\alpha_\lambda, \beta_\lambda) \qquad \text{(shape, rate)}$$
$$p \sim \text{BETA}(\alpha_p, \beta_p)$$

Full posterior distribution

$$[N_i, p, \lambda \mid Y_{ij}] \propto [Y_{ij} \mid N_i, p][N_i \mid \lambda][\lambda][p]$$

Full conditional distribution

$$[N_i \mid \cdot] \propto [Y_{ij} \mid N_i, p][N_i \mid \lambda]$$

$$\propto \prod_{i=1}^{n} \prod_{j=1}^{J} \frac{\cancel{N_i!}}{\cancel{y_{ij}!}(N_i - y_{ij})!} \cancel{p^{y_{ij}}} (1-p)^{N_i - y_{ij}} \frac{e^{-\lambda} \lambda^{N_i}}{\cancel{N_i!}}$$

$$\propto \prod_{i=1}^{n} \prod_{j=1}^{J} \frac{(\lambda(1-p))^{N_i - y_{ij}} e^{-(\lambda(1-p))}}{(N_i - y_{ij})!}$$

$$\propto \prod_{i=1}^{n} \frac{(\lambda(1-p))^{JN_i - \sum_{j=1}^{J} y_{ij}} e^{-(\lambda(1-p))}}{\prod_{j=1}^{J} (N_i - y_{ij})!}$$

Use MH!

$\square$

N-mixture model

Full conditional distributions continued

— Perry Williams

$$[p\,|\,\cdot] \propto [y_{ij}\,|\,N_i, p]\,[p]$$

$$\propto \prod_{i=1}^{n}\prod_{j=1}^{J} \cancel{\frac{N_i!}{y_{ij}!(N_i-y_{ij})!}}\, p^{y_{ij}}(1-p)^{N_i-y_{ij}}\; p^{\alpha_p-1}(1-p)^{\beta_p-1}$$

$$\propto \prod_{i=1}^{n}\prod_{j=1}^{J}\left\{ p^{y_{ij}}(1-p)^{N_i-y_{ij}}\; p^{\alpha_p-1}(1-p)^{\beta_p-1} \right\}$$

$$\propto p^{\sum_{i=1}^{n}\sum_{j=1}^{J} y_{ij}+\alpha_p-1}\,(1-p)^{\sum_{i=1}^{n}\left(JN_i-\sum_{j=1}^{J} y_{ij}\right)+\beta_p-1}$$

$$= \text{Beta}\left(\sum_{i=1}^{n}\sum_{j=1}^{J} y_{ij}+\alpha_p,\ \sum_{i=1}^{n}\left(JN_i-\sum_{j=1}^{J} y_{ij}\right)+\beta_p\right) \square$$

N- mixture model

Full conditional distribution continued

— Perry Williams

$$[\lambda \mid \cdot] \propto [N_i \mid \lambda][\lambda]$$

$$\propto \prod_{i=1}^{n} \left\{ \frac{e^{-\lambda} \lambda^{N_i}}{\cancel{N_i!}} \right\} \frac{\cancel{\beta_\lambda^{\alpha_\lambda}} \lambda^{\alpha_\lambda - 1} e^{-\beta_\lambda \lambda}}{\cancel{\Gamma(\alpha_\lambda)}}$$

$$\propto \prod_{i=1}^{n} \left\{ \lambda^{N_i} e^{-\lambda} \right\} \lambda^{\alpha_\lambda - 1} e^{-\beta_\lambda \lambda}$$

$$\propto \lambda^{\sum_{i=1}^{n} N_i + \alpha_\lambda - 1} e^{-(n + \beta_\lambda)\lambda}$$

$$= \text{GAMMA}\Big(\underbrace{\sum_{i=1}^{n} N_i + \alpha_\lambda}_{\text{shape}}, \underbrace{n + \beta_\lambda}_{\text{rate}}\Big) \qquad \square$$